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## Cosmological 3-Brane Solutions

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### Abstract

We analyze cosmological equations in the brane world scenario with one extra space-like dimension. We demonstrate that the cosmological equations can be reduced to the usual 4D Friedmann type if the bulk energy-momentum tensor is different from zero. We then generalize these equations to the case of a brane of finite thickness. We also demonstrate that when the bulk energy-momentum tensor is different from zero, the extra space-like dimension can be compactified with a single brane and show that the stability of the radius of compactification implies standard cosmology and vice versa. For a brane of finite thickness, we provide a solution such that the 4D Planck scale is related to the fundamental scale by the thickness of the brane. In this case, compactification of the extra dimension is unnecessary.

# 1 Introduction

Motivated by the discovery of M-theory [1, 2], there has been a tremendous increase of interest in a class of models (scenarios) with gravity and observable matter placed in a different number of space-like dimensions. This idea presents us with the enticing possibility to explain some long-standing particle physics problems by geometrical means. In M-theory, the correspondence of different string theories (through dualities) is achieved in an 11D framework. Generally, from a cosmological point of view, the scale factor of the 11th dimension can be related to the expectation value of the string dilaton. In the particular case of the dimensional reduction to heterotic string theory, where the hidden and matter sectors lie on separate 10D branes, the size of the compact 11th dimension is relatively large,  $r_{11} > M_P$ , so that one can relate the fundamental Planck scale with that of the GUT scale.

Taking this idea one step further, there are now several scenarios which try to relate the electroweak scale and the masses of observed particles with the fundamental higher dimensional Plank scale exploiting either large extra space-like dimensions [3, 4, 5] or an exponential scaling of the “warp” factor in extra dimensions [6]. We note that warp factors in a context of M-theory have been discussed earlier in [1, 7]. Both approaches assume that the SM particles are confined to a 3+1-dimensional slice (“brane”) of the  $n+3+1$ -dimensional space-time. Gravity is assumed to exist in the full higher dimensional space, “bulk”. Related static domain-wall solutions in  $N = 1$  supergravity were considered in [8]. In the first proposal [4], the fundamental gravitational scale  $M_*$  can be related to the usual 4D Plank scale via a volume factor,  $M_{\text{Pl}}^2 = M_*^{n+2} r^n$ , where  $r^n$  is the volume of the compact space. If  $r$  is sufficiently large,  $M_*$  can be as low as 1 TeV, thus providing a possible explanation to the gauge hierarchy problem. In the second proposal [6], one posits the configuration of a “gravitational condenser”. Two branes of opposite tensions, which gravitationally repel each other, are stabilized due to the negative cosmological constant in the bulk. As a result, the distance scales on the brane with negative tension are exponentially smaller than those on the positive tension brane, which can also explain why  $M_{\text{Pl}} \gg M_W$ .

The possibility that there exists an extra dimension or dimensions allows for a host of non-trivial phenomenology including the production of Kaluza-Klein excitations of gravitons at future colliders or their detection in high-precision measurements at low energies. Moreover, they drastically change the cosmology of early universe [9, 10], sometimes too drastically to be consistent with the observable world. Thus, for example, a 1 TeV-scale gravity scenario excludes a universe hotter than  $\sim 1$  GeV due to an enormous emission of bulk modes at higher temperatures [5].

Another serious problem emphasized recently is an unusual form of the Friedmann equations for the case of one extra dimension [10, 11, 12] which leads to a rather peculiar

behavior of the Hubble parameter for the matter on the brane

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho^2}{36M_*^6}. \quad (1)$$

This solution, subsequently checked in [14, 15, 16], suggests that the gravitational law no longer reduces to Newton's law on the 3-brane. While mathematically correct as a solution to Einstein's equations, this behavior, if nothing else, indicates that some key ingredient is missing if this theory is to be capable of describing our Universe. It is worth noting that this unusual behavior of  $H$  does not depend on the size of the extra dimension, but holds even for  $r \sim M_*^{-1} \sim M_{\text{Pl}}^{-1}$ . If the solution (1) is generic, it poses a most serious threat to the viability of "brane world" scenarios.

Therefore, it is important to review the assumptions under which the solution (1) was obtained. To begin with, it is generally assumed that the size of the extra dimension is fixed in time. Since, as was noted earlier, this amounts to fixing the vev of the dilaton, this is a reasonable assumption. A rolling dilaton implies the variation of gauge couplings and particle masses and, as with the Planck scale, there are very strong constraints against this [17]. In addition, there are several other assumptions which go into the derivation of (1), which we list as:

1. The cosmological constants on the wall and in the bulk vanish, ie.  $\Lambda_w, \Lambda_b = 0$ .
2. The brane is of zero thickness,  $\Delta=0$ .
3. The emptiness of the bulk,  $T_{\mu\nu}^{bulk} = 0$ .

While all of these assumptions are plausible, none are imperative. In particular, it is quite reasonable to expect the brane to have some finite thickness on the order of the fundamental scale or larger,  $\Delta > M_*^{-1}$ . The emptiness of the bulk,  $T_{\mu\nu}^{bulk} = 0$ , cannot be a generic property either. Interactions between the brane and gravity as well as other bulk mechanisms that might be responsible for the stabilization of radii should lead to a non-zero energy-momentum tensor in the bulk. Finally, the condition on the vanishing of the cosmological constants can certainly be lifted as the brane can have its own energy density, not related to the observable matter density  $\rho$ .

The case of nonvanishing  $\Lambda_w$  and  $\Lambda_b$  was studied by several groups [13, 14, 15, 16]. The use of the two brane construction [6] allows one to obtain the correct linear dependence of  $H^2$  with  $\rho$ . Instead of  $\rho^2$  in (1), one would have  $(\Lambda_w + \rho)^2 = \Lambda_w^2 + 2\Lambda_w\rho + \rho^2$  with  $\Lambda_w^2$  term canceled by negative  $\Lambda_b$ . As we will show, the introduction of two branes with  $\pm\Lambda_w$  tensions and bulk cosmological constant is not the unique way of recovering the conventional Friedmann equations.

In this letter we carefully analyze Einstein's equations in 4+1-dimensions with matter confined to a three-dimensional brane. In particular, we will derive sufficient conditions which ensure a smooth transition to conventional cosmology and Newton's law on the brane. To do this we relax the assumptions that  $\Delta = 0$  and  $T_{\mu\nu}^{bulk} = 0$ . If  $T_{\mu\nu}^{bulk} \neq 0$ , the transition to the conventional cosmology can be obtained with or without a vanishing  $\Delta$ . We further demonstrate that the extra space-like dimension in the thin-wall approximation can be compactified for a single brane and show that the stability of the radius of compactification implies standard cosmology and vice versa. We then generalize this result to the case of a finite wall thickness and give a specific solution in this case as well. Nevertheless, we find that finite  $\Delta$  and vanishing bulk energy-momentum tensor still lead to a phenomenologically unacceptable solution.

## 2 The Theoretical Framework

We start our analysis by considering the following 5-dimensional theory describing the coupling of the matter content of the universe with gravity

$$S = \int d^5x \sqrt{-\hat{g}} \left\{ \frac{M_5^3}{16\pi} \hat{R} + \hat{\mathcal{L}}_o \right\}, \quad (2)$$

where  $M_5$  is the fundamental five dimensional Planck mass and the hat denotes 5-dimensional quantities.  $\hat{\mathcal{L}}_o$  represents all other contributions to the action which are not strictly gravitational. These include the brane itself, matter on the brane, as well as any interaction between the brane and the bulk. As 5D Poincaré invariance is broken by the brane, we require only that  $\hat{\mathcal{L}}_o$  respect the surviving 4D Poincaré invariance. We also consider the following ansatz for the line-element of the 5-dimensional manifold

$$ds^2 = -n^2(t, y) dt^2 + a^2(t, y) \delta_{ij} dx^i dx^j + b^2(t, y) dy^2, \quad (3)$$

where  $\{t, x^i\}$  and  $y$  denote the 4-dimensional spacetime (in the direction of the brane) and the extra dimension, respectively.

The variation of the action functional (2) with respect to the 5-dimensional metric tensor  $\hat{g}_{MN}$  leads to the Einstein's equations which for the above spacetime background take the form (see e.g. [11, 12])

$$\hat{G}_{00} = 3 \left\{ \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) - \frac{n^2}{b^2} \left[ \frac{a''}{a} + \frac{a'}{a} \left( \frac{a'}{a} - \frac{b'}{b} \right) \right] \right\} = \hat{\kappa}^2 \hat{T}_{00}, \quad (4)$$

$$\begin{aligned}\hat{G}_{ii} &= \frac{a^2}{b^2} \left\{ \frac{a'}{a} \left( \frac{a'}{a} + 2\frac{n'}{n} \right) - \frac{b'}{b} \left( \frac{n'}{n} + 2\frac{a'}{a} \right) + 2\frac{a''}{a} + \frac{n''}{n} \right\} \\ &+ \frac{a^2}{n^2} \left\{ \frac{\dot{a}}{a} \left( -\frac{\dot{a}}{a} + 2\frac{\dot{n}}{n} \right) - 2\frac{\ddot{a}}{a} + \frac{\dot{b}}{b} \left( -2\frac{\dot{a}}{a} + \frac{\dot{n}}{n} \right) - \frac{\ddot{b}}{b} \right\} = \hat{\kappa}^2 \hat{T}_{ii},\end{aligned}\quad (5)$$

$$\hat{G}_{05} = 3 \left( \frac{n'}{n} \frac{\dot{a}}{a} + \frac{a'}{a} \frac{\dot{b}}{b} - \frac{\dot{a}'}{a} \right) = 0, \quad (6)$$

$$\hat{G}_{55} = 3 \left\{ \frac{a'}{a} \left( \frac{a'}{a} + \frac{n'}{n} \right) - \frac{b^2}{n^2} \left[ \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) + \frac{\ddot{a}}{a} \right] \right\} = \hat{\kappa}^2 \hat{T}_{55}, \quad (7)$$

where  $\hat{\kappa}^2 = 8\pi\hat{G} = 8\pi/M_5^3$ . Note that, in the above relations, the dots and primes denote differentiation with respect to  $t$  and  $y$ , respectively.

In this paper, we assume that the scale factor of the fifth dimension depends neither on space nor time, i.e.  $b = \text{const}$ . In that case, the (05)-component of Einstein's equations can be integrated to give the result

$$n(t, y) = \lambda(t) \dot{a}(t, y). \quad (8)$$

while the (00)-component reduces to a differential equation for  $a$  with respect to  $y$  with the general solution depending on the form of the energy density  $\hat{\rho}$  of the universe. For future reference, note that, by choosing the normalization  $n(t, y = 0) = 1$ , the Hubble parameter can be expressed in terms of  $\lambda(t)$  in the following way

$$H^2 \equiv \left( \frac{\dot{a}_0}{a_0} \right)^2 = \frac{1}{\lambda^2(t) a_0^2(t)}, \quad (9)$$

where the subscript 0 denotes quantities evaluated at  $y = 0$ .

In [12], it was assumed that the usual matter content of our universe is confined to a 4-dimensional hypersurface located at  $y = 0$ . In this case, the energy-momentum tensor of our brane-universe can be expressed in the form

$$T^A_B = \frac{\delta(y)}{b} \text{diag}(-\rho, p, p, p, 0) \quad (10)$$

The inhomogeneity in the distribution of matter in the 5-dimensional spacetime leads to the discontinuity of the first derivative of the metric tensor with respect to  $y$  and, thus, to the appearance of a Dirac delta function in its second derivative. By matching the coefficients of the delta functions that appear at both sides of the (00) and (ii) components of Einstein's

equations, the *jumps* in the first derivatives of  $a$  and  $n$  were derived and found to be

$$[a'] = a'(0^+) - a'(0^-) = -\frac{\hat{\kappa}^2}{3} \rho a_0 b_0, \quad (11)$$

$$[n'] = n'(0^+) - n'(0^-) = \frac{\hat{\kappa}^2}{3} (3p + 2\rho) n_0 b_0. \quad (12)$$

For cosmological solutions which are symmetric under the change  $y \rightarrow -y$  and for an empty bulk-universe, the (55) component of Einstein's equations (7) takes the form

$$\frac{\dot{a}_0^2}{a_0^2} + \frac{\ddot{a}_0}{a_0} = -\frac{\hat{\kappa}^4}{36} \rho (\rho + 3p). \quad (13)$$

The above equation leads to the 5-dimensional analog of the Friedmann equation given in eq. (1). However, as noted earlier, instead of the usual  $H^2 \propto \rho$  dependence, one finds  $H \propto \rho$  implying a departure from the standard cosmological expansion which would follow from the Newtonian force law. In subsequent work [18], we will show the precise effect on the gravitational force law which the various brane solutions imply.

In this class of exact cosmological solutions with  $b = \text{const}$ , the spatial scale factor  $a$  decreases as one moves away from our brane-universe at  $y = 0$ . The global definition of this group of *special* solutions throughout spacetime strongly relies on the existence of a second brane at  $y = 1/2$  whose matter content is heavily constrained by the energy density and pressure of the matter on our brane.

Here, we propose an alternative mechanism to restore the usual form of the Friedmann equation, in the framework of a 5-dimensional gravitational theory, which does not necessitate the introduction of a second brane or a cosmological constant. We will follow two different approaches: we first allow a non-vanishing energy-momentum tensor in bulk and subsequently consider the case of a brane of finite thickness,  $\Delta$ . The bulk energy-momentum tensor is given by

$$\hat{T}_B^A = \text{diag}(-\hat{\rho}, \hat{p}, \hat{p}, \hat{p}, \hat{T}_5^5), \quad (14)$$

which, when combined with the equation for the conservation of the energy-momentum tensor,  $D_M \hat{T}_N^M = 0$  leads to the following relations

$$\frac{d\hat{\rho}}{dt} + 3(\hat{\rho} + \hat{p}) \frac{\dot{a}}{a} + (\hat{\rho} + \hat{T}_5^5) \frac{\dot{b}}{b} = 0, \quad (15)$$

$$(\hat{T}_5^5)' + \hat{T}_5^5 \left( \frac{n'}{n} + 3 \frac{a'}{a} \right) + \frac{n'}{n} \hat{\rho} - 3 \frac{a'}{a} \hat{p} = 0. \quad (16)$$

In the following sections, we will derive the explicit form for the bulk energy-momentum tensor and the corresponding metric, which is consistent with the above energy conservation equations as well as the gravitational equations of motion (4-7).

### 3 Thin Wall Approximation

In our first approach, we adopt the concept of a brane-universe with zero thickness while allowing for a non-zero value of  $\hat{T}_5^5$  in the bulk. To do so, we assume that the energy momentum tensor on the brane has the form

$$\hat{T}_B^A = \text{diag}\left(\frac{\delta(y)}{b}(-\rho, p, p, p), T_5^5\right), \quad (17)$$

while outside the brane,  $\hat{\rho} = \hat{p} = 0$ , but  $\hat{T}_5^5$  retains a non-zero value consistent with the energy conservation equations. For a constant scale factor along the extra dimension, the (00)-component of Einstein's equations can be easily integrated and gives the following general solution for  $a$

$$a^2(t, y) = a_0^2(t) + c(t) |y| + \frac{b^2}{\lambda^2(t)} y^2, \quad (18)$$

outside the brane. The unknown function  $c(t)$  can be determined from the *jump* of the first derivative of  $a$ , given by eq.(11), and is found to be

$$c(t) = -\frac{\hat{\kappa}^2}{3} \rho a_0^2 b. \quad (19)$$

The general solution (18) for  $a^2$  has always a minimum at

$$y_{min}(t) = \pm \frac{c(t)\lambda^2(t)}{2b^2} = \pm \frac{\hat{\kappa}^2 \rho}{6b(\dot{a}_0/a_0)^2} \quad (20)$$

which in turn, by making use of eq. (9), leads to the evolution equation for  $a_0$

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{\hat{\kappa}^2 \rho}{3(2b|y_{min}(t)|)}. \quad (21)$$

From the above equation, one can immediately conclude that the Friedmann equation with the correct linear dependence on the density  $\rho$  is recovered if, and only if,  $y_{min}(t) = \text{const.}$  In that case, we can identify

$$\kappa^2 = \frac{\hat{\kappa}^2}{2b|y_{min}|} \Rightarrow M_P^2 = M_5^3 (2b|y_{min}|). \quad (22)$$

As we will, now, demonstrate, the time-independence of  $y_{min}$  and, thus, the restoration of the Friedmann equation strongly relies on a non-vanishing  $\hat{T}_5^5$  in the bulk. Indeed, by substituting the solution (18) for  $a$  in the (55)-component of Einstein's equations, we obtain the constraint

$$c(t) \dot{c}(t) - \frac{4b^2}{\lambda^2(t)} \dot{a}_0(t) a_0(t) + \frac{4b^2}{\lambda^3(t)} \dot{\lambda}(t) a_0^2(t) = \frac{\hat{\kappa}^2}{3} 4b^2 a^3(t, y) \dot{a}(t, y) \hat{T}_5^5, \quad (23)$$

which, when integrated with respect to time, leads to the following expression

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 \equiv \frac{1}{\lambda^2 a_0^2} = \frac{\hat{\kappa}^4 \rho^2}{36} - \frac{2\hat{\kappa}^2}{3a_0^4} \int a^3(t, y) \dot{a}(t, y) \hat{T}_5^5 dt, \quad (24)$$

for the Friedmann equation. From the above result, we can easily see that, if we choose  $\hat{T}_5^5 = 0$ , we recover the behavior  $H \sim \rho$ , which is characteristic of the case of an empty bulk-universe [12]. Clearly, a non-vanishing value for  $\hat{T}_5^5$  is required to recover the standard Friedmann expansion.

Any expression for  $\hat{T}_5^5 \neq 0$  must be consistent with the general conservation of  $\hat{T}_N^M$  and in particular the fifth component of those equations (16). For vanishing  $\hat{\rho}$  and  $\hat{p}$  in the bulk, the general solution of the aforementioned component takes the form

$$\hat{T}_5^5 = \frac{w(t)}{n(t, y) a^3(t, y)}. \quad (25)$$

The above expression allows us to make a suitable choice for  $\hat{T}_5^5$  that will cancel the quadratic dependence on  $\rho$  in (24). Indeed, we find that the following non-vanishing bulk value of  $\hat{T}_5^5$

$$\hat{T}_5^5 = -\frac{a_0^3(t)}{2n(t, y)a^3(t, y)} \left[ \frac{(\rho - 3p)}{2b|y_{min}|} + \frac{\hat{\kappa}^2}{6} \rho (\rho + 3p) \right] \quad (26)$$

leads to the time-independence of  $y_{min}$  and to the restoration of the Friedmann equation (21), at the same time. Given the fact that  $\rho$  and  $p$ , being defined at the origin, are functions of time only, the ansatz (26) for  $\hat{T}_5^5$  obviously belongs to the class of solutions given by (25).

In eqs. (22) and (26),  $2by_{min}$  represents the length scale that determines the 4-dimensional Planck mass  $M_P$  and the physical distance over which  $\hat{T}_5^5$  is smoothly distributed in the bulk. Moreover, this length can have another important interpretation. It can be shown that both  $a^2$  and  $n^2$  reach their extrema at the same point  $y = y_{min}$ . If one identifies the points  $y = |y_{min}|$  and  $y = -|y_{min}|$ , the extra dimension is effectively compactified with the size of the compact dimension being given by  $2by_{min}$ .

Before concluding this section, we would like to stress the importance of the bulk energy-momentum tensor. While one can certainly construct configurations (D-branes, solitons in some exotic field theory, etc.) where there is no bulk energy-momentum tensor (or cosmological constant in the bulk), in the thin wall case, there is no newtonian limit of gravity and such a model can be safely discarded at once. Instead, we have shown that one *must* have a non-vanishing bulk energy-momentum tensor or require at least two branes with opposite cosmological constants in addition to a bulk cosmological constant.

## 4 Thick Wall Approximation

In the second approach, we assume that our brane-universe has a non-vanishing thickness  $2\Delta$  and the energy density  $\hat{\rho}$  is homogeneously distributed over the 5-dimensional spacetime



of our brane. For zero pressure on the brane, the zeroth component (15) of the equation for the conservation of energy gives  $\hat{\rho} = \hat{\rho}_0/a^3$ , where  $\hat{\rho}_0$  is a constant both in  $t$  and  $y$ . In that case, the general solution for the spatial scale factor  $a$ , inside the brane, takes the form

$$\begin{aligned} \frac{2A^2}{B^3(t)} \log\left(\frac{2}{B(t)} [B^2(t)a(t, y) - A^2] + 2\sqrt{E_{in}(t) + B^2(t)a^2(t, y) - 2A^2a(t, y)}\right) \\ + \frac{2}{B^2(t)} \sqrt{E_{in}(t) + B^2(t)a^2(t, y) - 2A^2a(t, y)} = \pm\sqrt{2} [|y| + C_{in}(t)], \end{aligned} \quad (27)$$

where

$$A^2 = \frac{2b^2\hat{\kappa}^2\hat{\rho}_0}{3}, \quad B^2(t) = \frac{2b^2}{\lambda^2(t)} \quad (28)$$

and  $E_{in}(t)$  and  $C_{in}(t)$  are unknown functions of time which need to be determined. The  $\pm$  sign at the r.h.s. of eq. (27) corresponds to the sign of the first derivative of  $a$  with respect to  $y$ , inside the brane, which causes the scale factor to increase or decrease, respectively, as we move away from the origin. The symmetry of our cosmological solution, exhibited when  $y \rightarrow -y$ , leads to the vanishing of  $a'$  at the origin or equivalently to the condition

$$E_{in}(t) = 2A^2a_0(t) - B^2(t)a_0^2(t). \quad (29)$$

The function  $C_{in}(t)$  can be determined by evaluating the solution (27) at  $y = 0$ , and is found to be

$$C_{in}(t) = \pm \frac{\sqrt{2}A^2}{B^3(t)} \log\left(\frac{2}{B(t)} [B^2(t)a_0(t) - A^2]\right). \quad (30)$$

In order to determine  $E_{in}(t)$ , we substitute the implicit solution for  $a$  (27) in the remaining components of Einstein's equations. While the  $(ii)$  component is trivially satisfied, the  $(55)$  component leads to the following constraint

$$\frac{dE_{in}(t)}{dt} = \frac{4b^2}{3} \hat{\kappa}^2 a^3(t, y) \dot{a}(t, y) (\hat{\rho} + \hat{T}_5^5). \quad (31)$$

The fifth component of the conservation of the energy-momentum tensor  $\hat{T}_N^M$  (16) imposes the existence of a non-vanishing  $\hat{T}_5^5$  on the brane. Moreover, the choice  $\hat{T}_5^5 = 0$  would lead to an inconsistency in the above equation since  $E_{in}$  is, by definition, a function of time only.

The results of the previous section show that a non-vanishing bulk  $\hat{T}_5^5$  gives us the extra degree of freedom to recover the Friedmann equation on the brane. For the thick wall solution, we will demonstrate that, once again, the appropriate choice of  $\hat{T}_5^5$  lead to the usual dependence of the Hubble parameter on the energy density. As before, although the usual matter content should be localized on the brane-universe, the fifth component of the energy-momentum tensor  $\hat{T}_5^5$  must be smoothly distributed over the entire extra dimension. From the condition (29), we can easily see that the usual form of the Friedmann equation

can be obtained for  $E_{in} = 0$ . This is consistent with eq. (31) only if  $\hat{T}_5^5$  is exactly equal and opposite to the energy density  $\hat{\rho}$  of our brane-universe. In that case, we obtain

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{\kappa^2 \rho(t, 0)}{3}, \quad (32)$$

where we have used the fact that  $\hat{\rho} = \rho/(2\Delta b)$  and defined

$$\kappa^2 = \frac{\hat{\kappa}^2}{\Delta b} \Rightarrow M_P^2 = M_5^3 (\Delta b). \quad (33)$$

Note that the above expression for the Friedmann equation involves the 4-dimensional energy density  $\rho$  and gravitational constant  $\kappa^2$ , in distinction to the result (13) where the square of the 4-dimensional energy density was combined with the 5-dimensional gravitational constant  $\hat{\kappa}^2$ . Here, we want to note that the above choice,  $E_{in} = 0$ , corresponds only to a special solution of the system and that other solutions also leading to the restoration of the Friedmann equation do exist and will be discussed elsewhere [18].

By making use of the condition (29), for the choice  $E_{in} = 0$ , the implicit solution (27) for  $a$  inside the brane takes the form

$$\begin{aligned} \frac{a_0(t)}{B(t)} \log \left( B(t)[2a(t, y) - a_0(t)] + 2\sqrt{B^2(t)a(t, y)[a(t, y) - a_0(t)]} \right) \\ + 2\sqrt{\frac{a(t, y)}{B^2(t)}[a(t, y) - a_0(t)]} = \sqrt{2} \left( y + \frac{a_0(t)}{2B(t)} \log[B(t)a_0(t)] \right), \end{aligned} \quad (34)$$

from which it follows that  $a(t, y) \geq a_0(t)$  leading to the conclusion that the spatial derivative of the scale factor  $a$ , inside the brane, should be positive and, thus,  $a$  increases as we move away from the origin.

In order to complete the picture, we need to determine the solution for the scale factor  $a$  outside the brane. For vanishing  $\hat{\rho}$ , the general solution for  $a$  can be written as

$$a^2(t, y) = \frac{b^2}{\lambda^2(t)} [|y| + C_{out}(t)]^2 - \frac{\lambda^2(t)E_{out}(t)}{2b^2}, \quad (35)$$

where again  $C_{out}(t)$  and  $E_{out}(t)$  are functions of time which need to be determined. If we substitute the above solution for  $a$  in the (55)-component of Einstein's equations, we obtain a constraint for  $E_{out}(t)$  similar to eq. (31) but with  $\hat{\rho} = 0$ , while the (ii)-component is, again, trivially satisfied. By continuity, the value of  $\hat{T}_5^5$  outside the brane is given by

$$\hat{T}_5^5 = -\frac{\hat{\rho}_0 n(t, \Delta)}{n(t, y) a^3(t, y)}. \quad (36)$$

We can easily see that the above expression reduces to the inside value of  $\hat{T}_5^5$  in the limit  $y \rightarrow \Delta$  and is consistent with the general solution (24). The above expression, when substituted in the (55)-component, leads to the determination of  $E_{out}(t)$  while  $C_{out}(t)$  can be found by evaluating the solution (35) at  $y = \Delta$ . We may, then, write the final solution for the spatial scale factor outside the brane in the following way

$$a^2(t, y) = a_0(t)a(t, \Delta) + \frac{B^2(t)}{2} \left[ |y| - \Delta + \sqrt{\frac{2a(t, \Delta)}{B^2(t)} [a(t, \Delta) - a_0(t)]} \right]^2. \quad (37)$$

Here, we have to note that the solutions derived for  $a$ , and  $n$  through eq.(8), as well as their first derivatives with respect to  $y$ , are continuous across the wall, i.e. at  $y = \Delta$ , ensuring the smooth transition in spacetime from our matter dominated brane-universe to a  $\hat{T}_5^5$  dominated bulk-universe.

Finally, let us point out that the continuity of  $a'$  across the wall forces the spatial scale factor to increase as we move away from the brane in this particular solution. (There are of course other thick wall solutions where  $a$  decreases with  $y$ , as in the thin-wall solution. We will discuss these solutions elsewhere [18].) The absence of any minima for  $a$  outside the brane will cause the scale factor to increase indefinitely for a non-compact extra dimension. However, this is not a problem since the length scale that determines the 4-dimensional Planck scale  $M_P$ , in terms of  $M_5$ , is the thickness  $\Delta$  of the brane and not the size of the extra dimension.

It is instructive to analyze the last remaining possibility of the wall with a finite thickness and the vanishing  $\hat{T}_5^5$  in the bulk. As we noted earlier, directly from the 5th component of the energy conservation equation, we see that in the wall,  $\rho \neq 0$  implies that  $\hat{T}_5^5 \neq 0$ . If we assume  $\hat{T}_5^5 = 0$  in the bulk, we can use the continuity of  $\hat{T}_5^5$  across the wall and eq. (16) to find the solution for  $\hat{T}_5^5$  inside the wall:

$$\hat{T}_5^5 = \hat{\rho} \left[ \frac{n(t, \Delta)}{n(t, y)} - 1 \right]. \quad (38)$$

As before, we take  $\hat{\rho} = 0$ . Using this solution and the (55)-component of Einstein's equations, we obtain the evolution equation for  $a_0$ :

$$\left( \frac{\dot{a}_0}{a_0} \right)^2 + \frac{\ddot{a}_0}{a_0} = -\frac{\hat{\kappa}^2}{6b} \rho \frac{n(t, \Delta) - 1}{\Delta} \quad (39)$$

It is easy to see that the r.h.s of eq. (39) does not correspond to the usual form of the Friedmann equation. Thus, we conclude that the case of  $\Delta \neq 0$  and  $\hat{T}_5^5 = 0$  in the bulk does not lead to the standard cosmological expansion for the spatial scale factor.

## 5 Conclusions

The idea that our four-dimensional spacetime is a slice of a higher dimensional space time has always been an intriguing one. In most cases, one generally assumes that the physical size of the extra dimension is small (of order  $M_P^{-1}$ ) and we can decompose the higher-dimensional states in a Fourier expansion of momentum modes in the extra dimensions. This is the usual Kaluza-Klein decomposition. Of course there is no inherent reason that the size of the extra dimensions need be small (or even compact!). Our Universe as a three-brane however, is not a trivial proposition. As shown in several recent works, the cosmological solutions to Einstein's equations for a three-brane embedded in a higher dimensional space-time, leads to the unphysical solution that the Hubble parameter on the 3-brane is proportional to the matter density. This expansion law is not consistent with Newtonian gravity. This conundrum can be fixed by balancing the cosmological constants on the brane and in the bulk but requires two branes for a complete solution.

In this paper, we have derived alternative solutions for recovering the normal Hubble expansion on the three-brane. Our solutions do not require the existence of a second brane. In both types of solutions we have presented, we require that  $\hat{T}_5^5$  is non-vanishing in the bulk. Keeping to the thin wall approximation, we have shown that by an appropriate choice of  $\hat{T}_5^5$ , the 3-space scale factor which decreases as we move away from the brane, has a fixed minimum. As such, by identifying the points  $\pm y_{min}$ , we can compactify the extra dimension without the need of a second brane. The 4D Plank scale is then determined by the compactification radius,  $M_P^2 = M_5^3(2by_{min})$ . In the thick wall solution, the size of the extra dimension remains infinite, and the Planck scale is determined by the thickness of the wall,  $M_P^2 = M_5^3(2b\Delta)$ .

We also want to stress that in any realization of the world as a brane scenario in which there is a 4-dimensional Einstein gravity on a brane, one must have the normal law  $H^2 \sim \rho$  which simply follows from newtonian limit of general relativity. To do so, it appears that the bulk can not be ignored. Either there must exist a bulk cosmological constant or a non-vanishing  $\hat{T}_5^5$ .

In this paper, we have not specified any particular source which accounts for a non-vanishing  $\hat{T}_5^5$ . This alone is worthy of a separate investigation. However, we note that the normal 4D form of the Friedmann equations can be recovered if the value of  $\hat{T}_5^5$  on the brane is proportional to the trace of the 4D energy-momentum tensor. This bears a strong resemblance to the case of the dilaton interaction with matter, which is also proportional to  $T_\mu^\mu$  on the brane. However, we cannot claim that this solution is unique. In the case of the thin wall, where the extra dimension is compact, the existence of  $\hat{T}_5^5$  in the bulk should be related to the physics responsible for the stabilization of the extra dimension, and perhaps equivalently the stabilization of the dilaton vev. These and other related issues will be

addressed in a future publication [18].

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